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THE GRAVITATIONALLY REPULSIVE DOMAIN WALL

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Abstract:

The Gauss-Codazzi formalism is used to obtain exact solutions to Einstein's equations in the presence of domain walls. Domain walls are shown to have repulsive gravitational fields. The most general solution to Einstein's equations for a planar domain wall is obtained. Also, the motion of a spherical domain wall in an asymptotically flat space-time is derived.

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I. INTRODUCTION

Recent investigations in cosmology have focused in a special way on the role of particle physics in the evolution of the universe into its presently observed state. Particle theories suggest that phase transitions of various kinds occur in the early universe and produce important effects that, until fairly recently, had been largely ignored. In the "inflationary universe" cosmology [1], for example, an early phase transition fills the universe with a false vacuum for awhile, causing an exponential DeSitter-like expansion until the true vacuum takes over, and thereby providing a possible solution to the horizon and flatness problems of the standard cosmological model. Phase transitions can also give birth to soliton-like structures such as monopoles, strings and domain walls [2].

Within the context of general relativity, domain walls are immediately recognizable as especially unusual and interesting sources of gravity. As was pointed out by Zel'dovich, Kobzarev and Okun [3], the stress-energy of domain walls is composed of surface energy density and strong tension in two spatial directions, with the magnitude of the tension equal to that of the surface energy density. This is interesting because there are several indications that tension acts as a repulsive source of gravity in general relativity, whereas pressure is attractive. This is evident, for example, from the Raychaudhuri equation relating the expansion of geodesic congruences to the local stress-energy and from the way the pressure of spherical stellar models appears in the relativistic equations governing their structure [4]. It is also implied by the fact that a domain wall dominated universe expands like $R \sim t^2$, where R is the cosmological scale parameter and t is cosmological time [3]. The question thus arises whether domain walls exhibit <u>repulsive gravitational fields</u>, and, if yes, what are the implications thereof.

Vilenkin [5] addressed this question by linearizing Einstein's equations (weak field approximation) in the presence of a plane static wall. He found that the linearized equations admit static solutions, and that these do indeed correspond to repulsive gravitational fields. As Vilenkin pointed out however, the static solutions to the linearized equations do not properly match up to the known general exact static solution with planar symmetry, except for the special case $\tau = \frac{1}{4}\sigma$, where τ and σ are respectively the tension and the surface energy density of the wall (for a domain wall, $\tau = \sigma$). It appears that, unless $\tau = \frac{1}{4}\sigma$, the weak field approximation of Einstein's equations has static solutions to which correspond no static solution of the exact Einstein equations. This results in some uncertainty as to how to interpret the solutions to the linearized equations.

In this paper, we address the questions raised above by finding exact solutions to Einstein's equations in the presence of domain walls. We use the Gauss-Codazzi formalism for describing the geometry of surfaces embedded in higher dimensional curved spaces (section II). From the Gauss-Codazzi equations we derive an unambiguous answer to the question whether and in what sense domain walls have repulsive gravitational fields: for an observer to ride along next to a domain wall, he must fire a rocket away from the wall (or use some other means to balance the wall's repulsion), or the wall must accelerate toward him, or both. In section III, we derive the motion of a spherical domain wall in an asymptotically flat space-time. We find that such a wall always collapses to a black hole, and that it is always attractive to a distant observer. This latter property is not inconsistent with the repulsive character of domain walls because the spherical domain wall accelerates towards its center, thereby increasing its separation from an initially co-moving external geodesic observer. In section IV, we consider "planar" walls, i.e. walls which are homogeneous and isotropic in their two space-like directions. We show that there are no reflection symmetric static solutions to Einstein's equations for a "planar" wall unless $\tau = \frac{1}{4}\sigma$. Next we derive the most general reflection symmetric solution to Einstein's equations for a "planar" domain wall ($\tau = \sigma$). We analyze in detail a particular solution which corresponds to a uniform gravitational field in which observers on either side are repelled by the wall with constant acceleration $2\pi G_N \sigma$, where G_N is Newton's gravitational constant. Finally, in section V, we discuss our results and draw some conclusions.

Throughout, we adopt the convention in which the space-time metric has signature $- + + +$.

II. THE GAUSS-CODAZZI FORMALISM

We wish to solve Einstein's equations in the presence of stress-energy sources confined to 3-dimensional time-like hypersurfaces (infinitesimally thin shells of stress-energy). Following Israel [6], we shall use the Gauss-Codazzi formalism, which greatly streamlines the formulation of the problem.

a. The equations

Let S denote a 3-dimensional time-like hypersurface containing stress-energy and let ξ^a be its unit space-like normal ($\xi_a \xi^a = +1$). The 3-metric intrinsic to the hypersurface S is

$$h_{ab} = g_{ab} - \xi_a \xi_b, \tag{2.1}$$

where g_{ab} is the 4-metric of space-time. Let ∇_a denote the covariant derivative associated with g_{ab} and let

$$D_a = h_a{}^b \nabla_b. \tag{2.2}$$

The extrinsic curvature π_{ab} of S is then defined by

$$\pi_{ab} \equiv D_a \xi_b = \pi_{ba}. \tag{2.3}$$

In terms of the extrinsic curvature, the contracted forms of the first and second Gauss-Codazzi equations are

$${}^3R + \pi_{ab}\pi^{ab} - \pi^2 = -2\, G_{ab}\xi^a \xi^b \tag{2.4a}$$

$$h_{ab} D_c \pi^{bc} - D_a \pi = G_{bc} h^b{}_a \xi^c. \tag{2.4b}$$

Here ^3R is the Ricci scalar curvature of the 3-geometry h_{ab} of S, $\pi = \pi_a{}^a$, and $G_a{}^b$ is the Einstein tensor in 4-dimensional space-time.

In the situations of interest to us, the stress-energy tensor T_{ab} of 4-dimensional space-time has a delta-function singularity on S. This implies that the extrinsic curvature π_{ab} has a jump discontinuity across S, since π_{ab} is analogous to the gradient of the Newtonian gravitational potential. Hence one is led to introduce on S

$$\gamma_{ab} \equiv \pi_{+ab} - \pi_{-ab} \tag{2.5a}$$

and

$$S_{ab} \equiv \int d\ell \; T_{ab} \; , \tag{2.5b}$$

where ℓ is the proper distance through S in the direction of the normal ξ^a, and where the subscripts $\pm$ refer to values just off the surface on the side determined by the direction of $\pm\,\xi^a$. Using Einstein's and the Gauss-Codazzi equations, one can show that [cf. refs. (4) and (6)]

$$S_{ab} = \frac{-1}{8\pi G_N}\,(\gamma_{ab} - h_{ab}\gamma_c{}^c). \tag{2.6}$$

One also introduces

$$\hat{\pi}_{ab} = \frac{1}{2}\,[\pi_{+ab} + \pi_{-ab}]. \tag{2.7}$$

Then, by virtue of Eq. (2.6), the sums and differences of Eq. (2.4) on opposite sides of S yield in vacuo (i.e. if T_{ab} vanishes off S)

$$h_{ac} D_b S^{cb} = 0, \tag{2.8a}$$

$$h_{ac} D_b \hat{\pi}^{cb} - D_a \hat{\pi} = 0, \tag{2.8b}$$

$$\hat{\pi}_{ab} S^{ab} = 0, \tag{2.8c}$$

$$^3R + (\hat{\pi}_{ab}\hat{\pi}^{ab} - \hat{\pi}^2) = - 16\, \pi^2 G_N^2 \left(S_{ab}S^{ab} - \frac{1}{2}\,(S_a{}^a)^2\right) . \tag{2.8d}$$

The following form a complete set of equations to solve Einstein's equations in the presence of a thin wall:

- Einstein's equations off S.
- Eqs. (2.6) and (2.8), and continuity of the metric g_{ab} across S.
- A suitable description of the matter on S such as Eq. (2.9) below augmented by an equation of state.
- Initial data.

b. The Surface Stress-Energy Tensor

We shall restrict our attention to sources for which

$$S^{ab} = \sigma\, u^a u^b - \tau\, (h^{ab} + u^a u^b), \tag{2.9}$$

where u^a is the 4-velocity of any observer whose world line lies within S and who sees no energy flux in his local frame, and where σ and τ are respectively the surface energy-density (energy per unit area) and tension measured by that observer. For a dust wall, $\tau = 0$. For a domain wall $\tau = \sigma$ and hence

$$S^{ab} = - \sigma\, h^{ab}. \tag{2.10}$$

Since $(h^{ab}) = \mathrm{diag}\,(-1, +1, +1, 0)$ in the local frame of any observer moving within the surface, it is clear that all such observers measure the same surface energy density and tension. Motion parallel to a (pure) domain wall is undetectable.

For the choice (2.9), the conservation equation (2.8a) becomes

$$(\sigma-\tau)h_{ac}u^bD_bu^c + u_aD_b[(\sigma-\tau)u^b] - h_a{}^bD_b\tau = 0. \tag{2.11}$$

The parts of this equation parallel to and perpendicular to u^a are, respectively,

$$D_b(\sigma u^b) - \tau\, D_bu^b = 0 \tag{2.12a}$$

and

$$(h_a{}^b + u_au^b)D_b\tau - (\sigma-\tau)h_{ac}u^bD_bu^c. \tag{2.12b}$$

It follows immediately that for domain walls ($\tau=\sigma$), σ is a constant, i.e. σ has the same value at all events on the 3-dimensional time-like surface S. Whereas for dust walls ($\tau=0$), we have $D_b(\sigma u^b) = 0$, which states that the total amount of dust is conserved.

c. Attractive Energy and Repulsive Tension

Particularly useful combinations of the equations of Section II.a involve the accelerations of observers who hover just off S on either side. Let the vector field u^a be extended off S in a smooth fashion. The acceleration

$$u^a \nabla_a u^b = (h^b{}_c + \xi^b \xi_c)\, u^a \nabla_a u^c$$

$$= h^b{}_c u^a D_a u^c - \xi^b u^a u^c \pi_{ac} \tag{2.13}$$

has a jump discontinuity across S since π_{ac} has such a discontinuity. The perpendicular components of the accelerations of observers hovering just off S on either side satisfy [use Eq. (2.8c)]

$$\xi_b u^a \nabla_a u^b|_+ + \xi_b u^a \nabla_a u^b|_-$$

$$= -2u^a u^b \hat{\pi}_{ab} = -2\, \frac{\tau}{\sigma}\, (h^{ab} + u^a u^b)\hat{\pi}_{ab} \tag{2.14a}$$

and

$$\xi_b u^a \nabla_a u^b|_+ - \xi_b u^a \nabla_a u^b|_- = -u^a u^b \gamma_{ab}$$

$$= -4\pi G_N u^a u^b (h_{ab} S_p{}^p - 2\, S_{ab}) = 4\pi G_N (\sigma - 2\tau). \tag{2.14b}$$

The covariant equation (2.14b) states an interesting general result about the gravitational properties of walls. Consider first the case of a plane wall with reflection symmetry. In that case

$$\xi_b u^a \nabla_a u^b|_+ = -\xi_b u^a \nabla_a u^b|_- = 2\pi G_N (\sigma - 2\tau). \tag{2.15}$$

It follows that an observer who wishes to remain stationary next to the wall must accelerate away from the wall if $(\sigma - 2\tau) > 0$, and towards it if $(\sigma - 2\tau) < 0$. In this sense, a wall is attractive or repulsive depending on whether $\sigma-2\tau$ is positive or negative. To hover next to a dust wall, one should fire a jet engine whose thrust per unit mass is $2\pi G_N\sigma$ away from the wall. Whereas to hover next to a domain wall, one should fire a jet engine whose thrust per unit mass is $2\pi G_N\sigma$ towards the domain wall. Of course, if the wall itself is being accelerated, only the difference of the accelerations of the observers hovering on either side is a measure of the wall's gravitational pull or push. And that is precisely what Eq. (2.14b) expresses.

III. SPHERICAL WALLS

In this section we shall obtain the asymptotically flat solutions to Einstein's equations for spherically symmetric domain walls. For as long as possible, we shall proceed without specifying the relationship between σ and τ. Our analysis closely parallels that of Israel, who obtained the solutions for dust walls.

For a spherical shell of stress-energy, let the unit normal ξ^a point in the outward radial direction. It is well-known that asymptotic flatness and spherical symmetry require the exterior geometry to be Schwarzchildean, whereas the interior geometry is flat (Birkhoff's theorem). Hence

$$(ds^2)_+ = -e^{\nu(r)}dt^2 + e^{-\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad \text{for } r > R(t) \tag{3.1a}$$

and

$$(ds^2)_- = - dT^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad \text{for } r < R(t). \tag{3.1b}$$

Here M is the mass as measured by a distant external observer and

$$r = R(t) \tag{3.2}$$

is the equation of the wall. One finds for the components of u^a and ξ^a [a = t or T, r, θ, ϕ in that order]

$$(u^a{}_+) = (e^{-\nu}\beta, \dot{R}, 0, 0) \quad , \qquad (u^a{}_-) = (\alpha, \dot{R}, 0, 0)$$

$$(\xi^a{}_+) = (e^{-\nu}\dot{R},\beta,0,0) \quad , \qquad (\xi^a{}_-) = (\dot{R},\alpha,0,0) \ . \tag{3.3}$$

Here a dot denotes a derivative with respect to proper time of an observer moving with 4-velocity u^a at the wall, and

$$\alpha \equiv \dot{T} = (1 + \dot{R}^2)^{1/2}$$

$$\beta \equiv e^{\nu}\dot{t} = \left(1 - \frac{2G_N M}{R} + \dot{R}^2\right)^{1/2}. \tag{3.4}$$

These expressions and the definitions (2.2), (2.3) and (2.7) imply that

$$(h^{ab} + u^a u^b)\,\hat{\pi}_{ab} = (\xi_+{}^r + \xi_-{}^r)\,\frac{1}{R} \tag{3.5}$$

and

$$\xi_b u^a \nabla_a u^b|_+ = \frac{1}{\beta}\left(\ddot{R} + \frac{G_N M}{R^2}\right)$$

$$\xi_b u^a \nabla_a u^b|_- = \frac{1}{\alpha}\,\ddot{R} \quad . \tag{3.6}$$

Substitution into Eq. (2.14) then yields the equations of motion

$$(\alpha + \beta)\,\ddot{R} = -\frac{\alpha G_N M}{R^2} - \frac{2\tau}{\sigma}\,\frac{\alpha\beta(\alpha+\beta)}{R} \tag{3.7a}$$

$$(\alpha - \beta)\,\ddot{R} = -\frac{\alpha G_N M}{R^2} + 4\pi G_N(\sigma-2\tau)\alpha\beta \ . \tag{3.7b}$$

Equation (3.7a) tells us that $\ddot{R}$ is always negative if $\tau \geqslant 0$. Hence, a spherical wall with $\tau \geqslant 0$ always collapses to a black hole, regardless of its size. Another result is the expression for the mass

$$M = \frac{1}{2} \left[\sqrt{1+\dot{R}^2} + \sqrt{1 - \frac{2G_N M}{R} + \dot{R}^2} \right] 4\pi\sigma R^2 \qquad (3.8)$$

obtained by eliminating $\ddot{R}$ from Eq. (3.7). This expression implies that, independently of the value of τ, the mass M is positive definite(because σ is always positive) and hence a distant observer is always attracted by the spherical domain wall.

By Birkhoff's theorem, M is a constant. Consequently, if the conservation laws (2.12) can be solved for σ in terms of R and $\dot{R}$, then (3.8) yields a first integral of the equations of motion and the solution is obtainable by a quadrature. As we already noted in section IIb, the conservation laws are solved easily for domain walls ($\tau = \sigma$ =constant) and for dust walls ($\tau = 0$, σR^2 = constant). In these cases, one verifies quickly that Eqs. (3.7) guarantee that $\dot{M} = 0$; and further, that the remaining equations of the formalism are all automatically satisfied.

Equation (3.8) implies that

$$M = 4\pi\sigma\, R_m^2\, (1-2\pi\sigma G_N R_m) \qquad \text{for } R_m < \frac{1}{4\pi G_N \sigma}\,, \qquad (3.9)$$

where R_m is the maximum value of R, i.e. the value for which $\dot{R} = 0$. Our analysis breaks down for $R > \frac{1}{4\pi G_N \sigma}$, since a spherical shell that large is already within its Schwarzschild radius $R_{Sch} = 2G_N M$. Note that, for fixed σ, the ratio R_m/R_{Sch} decreases towards unity with increasing R_m.

Finally, in the present circumstance, note the sense in which spherical shells with $\tau/\sigma > 1/2$ exhibit repulsive characteristics. According to Eq. (3.6), both internal and external observers must accelerate inward in order to keep up with the collapsing shell when $\tau/\sigma > 1/2$ (this is obvious for the internal observer since he is in a flat region); but, of the two, the external observer must accelerate more strongly. If $\tau/\sigma < 1/2$, it is the internal observer who must accelerate more strongly.

IV. "PLANAR" WALLS

In this section we shall solve Einstein's equations for the gravitational fields of domain walls under the following symmetry conditions:

1. The wall is homogeneous and isotropic in its two space dimensions.
2. The space-time geometry is reflection symmetric with respect to the wall.

Under these conditions, we can find a coordinate system in which the wall is in the z = 0 "plane" and in which the geometry has the form

$$ds^2 = e^{2\nu(t,|z|)}(-dt^2 + dz^2) + B(t,|z|)\,(dx^2 + dy^2). \tag{4.1}$$

For lack of a better word, we refer to this case as that of the "Planar" Wall.

a. The Vacuum Equations and Their Solutions

The solutions to the vacuum Einstein equations for the metric (4.1) are obtained easily and are presumably known, as they are analogs of the Weyl solutions for axisymmetry. Substituting (4.1) into the definition of the Ricci tensor R_{ab} and setting $R_{ab} = 0$ yields

$$-B_{,tt} + B_{,zz} = 0 \tag{4.2a}$$

$$B_{,tz} - \frac{1}{2B} B_{,t} B_{,z} - B_{,t}\,\nu_{,z} - B_{,z}\,\nu_{,t} = 0 \tag{4.2b}$$

$$B_{,tt} + B_{,zz} - \frac{B_{,t}^{\;2} + B_{,z}^{\;2}}{2B} - 2B_{,t}\,\nu_{,t} - 2B_{,z}\,\nu_{,z} = 0 \tag{4.2c}$$

and

$$-\nu_{,tt} + \nu_{,zz} + \frac{B_{,t}^{\ 2}}{4B^2} - \frac{B_{,z}^{\ 2}}{4B^2} = 0 \ , \tag{4.2d}$$

where a comma subscript denotes differentiation with respect to the coordinates following it.

The most general reflection symmetric solution to Eq. (4.2a) is

$$B(t,|z|) = F(t - |z|) + G(t + |z|), \tag{4.3}$$

where F and G are arbitrary functions. Substitution of (4.3) into (4.2) yields for $z > 0$

$$F'' - \frac{F'^2}{2(F+G)} - F' \ (\nu_{,t} - \nu_{,z}) = 0, \tag{4.4a}$$

$$G'' - \frac{G'^2}{2(F+G)} - G' \ (\nu_{,t} + \nu_{,z}) = 0, \tag{4.4b}$$

and

$$-\nu_{,tt} + \nu_{,zz} = \frac{-F'G'}{(F+G)^2}, \tag{4.4c}$$

where a prime denotes differentiation of a function with respect to its argument. We separate the nontrivial solutions to equations (4.4) into two classes distinguished by the vanishing or nonvanishing of F' or G'.

i. <u>Class I: $F' = 0$ or $G' = 0$ (but not both)</u>

Suppose $G' = 0$. Then Eq. (4.4c) implies that

$$2\nu = \ell n \ H(t - z) + \ell n \ K(t + z) \tag{4.5}$$

for $z > 0$, where H and K are arbitrary; and Eq. (4.4a) then becomes

$$\frac{F''}{F'} - \frac{1}{2}\frac{F'}{F} - \frac{H'}{H} = 0, \tag{4.6}$$

which implies that

$$2\nu = \ln \left[\frac{F'(t-|z|)\ K(t+|z|)}{F^{1/2}(t-|z|)}\right] \tag{4.7}$$

for all z. In Eq. (4.7) we have used the reflection symmetry of the metric.

Similarly, if $F' = 0$,

$$2\nu = \ln \left[\frac{G'(t+|z|)\ H(t-|z|)}{G^{1/2}(t+|z|)}\right] . \tag{4.8}$$

ii. <u>Class II: $F' \neq 0$ and $G' \neq 0$.</u>

In this case, Eqs. (4.4a) and (4.4b) can be solved for $\nu_{,t}$ and $\nu_{,z}$:

$$2\nu_{,t} = \frac{F''}{F'} + \frac{G''}{G'} - \frac{1}{2}\frac{F'+G'}{F+G} \tag{4.9a}$$

$$2\nu_{,z} = -\frac{F''}{F'} + \frac{G''}{G'} - \frac{1}{2}\frac{G'-F'}{F+G} , \tag{4.9b}$$

for $z > 0$. Eq. (4.4c) is then automatically satisfied. Since, for $z > 0$, $F' = F_t = - F_z$, $G' = G_t = G_z$ and so forth, Eqs. (4.9) are readily integrated with the result

$$2\nu = \ln \left(C_0 \frac{F'(t-|z|)\ G'(t+|z|)}{[F(t-|z|) + G(t+|z|)]^{1/2}}\right) \tag{4.10}$$

for all z. C_0 is the constant of integration.

b. The Junction conditions at z = 0

The functions appearing in the above solutions are determined (up to initial conditions) by satisfying the Gauss-Codazzi junction conditions of section II at $z = 0$. The vectors u^a and ξ^a defined in section II have components ($a = t, z, x, y$)

$$(u^a) = (e^{-\nu},0,0,0), \quad (\xi^a) = (0,e^{-\nu},0,0) \tag{4.11}$$

in the coordinate system (4.1). The metric intrinsic to the wall at $z = 0$ is

$$\begin{aligned} h_{ab} &= g_{ab}\big|_{z=0} && \text{for } a, b \neq z \\ &= 0 && \text{for } a \text{ or } b = z; \end{aligned} \tag{4.12}$$

and the extrinsic curvature is

$$\begin{aligned} \pi_{ab}\big|_\pm = h_a{}^c h_b{}^d \nabla_c \xi_d\big|_\pm &= \tfrac{1}{2}\, g_{ab,z}\, e^{-\nu}\big|_\pm && \text{for } a, b \neq z \\ &= 0 && \text{for } a \text{ or } b = z\,. \end{aligned} \tag{4.13}$$

The junction conditions are Eqs. (2.6) and (2.8). Eqs. (2.8b) and (2.8c) are trivially satisfied since, by symmetry, $\hat{\pi}_{ab} = 0$. Eq. (2.6) yields

$$\begin{aligned} \gamma_{ab} &= g_{ab,z}\, e^{-\nu}\big|_+ = -\, g_{ab,z}\, e^{-\nu}\big|_- \\ &= 4\pi G_N(h_{ab} S_c{}^c - 2\, S_{ab}) \end{aligned}$$

$$= - 4\pi G_N[\sigma h_{ab} + 2 (\sigma - \tau) u_a u_b] \quad (a, b \neq z). \tag{4.14}$$

The (t,t) and (x,x) components of this equation are respectively

$$\nu_{,z}\Big|_+ = - \nu_{,z}\Big|_- = 2\pi G_N(\sigma - 2\tau)\, e^{\nu(t,0)} \tag{4.15a}$$

and

$$e^{\nu(t,0)} = \frac{1}{4\pi G_N\sigma} \frac{F'(t) - G'(t)}{F(t) + G(t)} . \tag{4.15b}$$

The perpendicular component (2.12b) of the conservation equation (2.8a) is vacuous as it should be, whereas the parallel component (2.12a) yields

$$\sigma \left[(F+G)\, e^{\nu}\right]^{1 - \frac{\tau}{\sigma}} \Big|_{z = 0} = C_1 \tag{4.16}$$

where C_1 is a constant of integration. To obtain (4.16) we have assumed τ/σ to be a constant as is the case for both dust walls and domain walls. Finally, Eq. (2.8d) yields

$$\frac{e^{-2\nu}}{F+G} \left\{(F'+G') \left[2\nu' + \frac{1}{2} \frac{F'+G'}{F+G}\right] -2(F''+G'')\right\}\Big|_{z = 0}$$

$$= 8\pi^2 G_N^2 \sigma(\sigma-4\tau), \tag{4.17}$$

where it is to be understood that $\nu'\Big|_{z = 0} = \frac{\partial \nu}{\partial t} (t, z = 0)$.

As an intermediate step in deriving Eq. (4.17) from Eq. (8d), one obtains the scalar curvature ^3R of the z = 0 wall,

$$^{3}R = e^{-2\nu}\left[\frac{2B''}{B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - 2\nu'\,\frac{B'}{B}\right]\Bigg|_{z\,=\,0}, \tag{4.18}$$

where B' and B" are defined analogously to ν'. Eq. (4.17) implies that a necessary condition for a static solution is $\tau = \frac{1}{4}\,\sigma$, a result promised in the Introduction. One readily verifies that for $\tau = \frac{1}{4}\,\sigma$ there is a static solution [5]

$$ds^2 = \frac{-dt^2 + dz^2}{\sqrt{1 + E|z|}} + (1 + E|z|)\,(dx^2 + dy^2) \tag{4.19}$$

that satisfies all equations (4.10), (4.15), (4.16) and (4.17) provided $E = -\,4\pi G_N\sigma$. This solution has the unique form appropriate for a static vacuum solution with planar symmetry [7].

c. <u>The Solutions for Domain Walls</u>

In this subsection we shall derive all "Planar" Domain Wall solutions.

i. <u>Class I: F' = 0 or G' = 0 (but not both).</u>

Suppose G' = 0 and $\tau = \sigma$. The equations we must solve are (4.7), (4.15), (4.16) and (4.17). Actually, Eq. (4.17) is implied by Eq. (4.15b). And Eq. (4.16) just tells us that σ = constant, something we already know to be generally true for domain walls. Combining Eqs. (4.7) and (4.15b) yields

$$K = \frac{1}{(4\pi G_N\sigma)^2}\,\frac{F'}{F^{3/2}}, \tag{4.20}$$

which assures that Eq. (4.15a) is satisfied as well. Consequently, the general class I domain wall solution for G' = 0 has non-vanishing metric components

$$g_{xx} = g_{yy} = F(t-|z|)$$

$$-g_{tt} = g_{zz} = \frac{1}{(4\pi G_N \sigma)^2} \frac{F'(t-|z|)\, F'(t+|z|)}{F^{1/2}(t-|z|)\, F^{3/2}(t+|z|)} , \tag{4.21}$$

where F is an arbitrary function determined by the initial data. Note that

$$F > 0 \quad \text{and} \quad F' > 0 \tag{4.22}$$

are required by Eq. (4.15b) and our demand that x,y,z are space-like coordinates whereas t is a time-like coordinate.

Similarly, the general class I domain wall solution for $F' = 0$ has non-vanishing metric components

$$g_{xx} = g_{yy} = G(t+|z|)$$

$$-g_{tt} = g_{zz} = \frac{1}{(4\pi G_N \sigma)^2} \frac{G'(t+|z|)\, G'(t-|z|)}{G^{1/2}(t+|z|)\, G^{3/2}(t-|z|)} , \tag{4.23}$$

with the requirement that $G > 0$ and $G' < 0$.

ii. Class II: $F' \neq 0$ and $G' \neq 0$.

We must solve Eqs. (4.10), (4.15), (4.16) and (4.17) for $\tau = \sigma$. Again Eq. (4.16) just tells us that σ = constant. One readily shows that Eqs. (4.10), (4.15a) and (4.15b) are equivalent to Eq. (4.10) and the following two equations

$$C_0 = \frac{1}{(4\pi G_N \sigma)^2} \frac{(F'-G')^2}{F'G'(F+G)^{3/2}} \tag{4.24}$$

$$\frac{F''}{F'} - \frac{G''}{G'} - \frac{3}{2}\frac{F'-G'}{F+G} = 0 , \tag{4.25}$$

and that Eq. (4.17) follows from Eqs. (4.10), (4.24) and (4.25). Differentiating (4.24) on both sides yields

$$0 = \frac{F'^2 - G'^2}{F'G'(F+G)^{3/2}} \left[\frac{F''}{F'} - \frac{G''}{G'} - \frac{3}{2}\,\frac{F'-G'}{F+G}\right]. \tag{4.26}$$

Consequently, Eq. (4.25) is implied by Eq. (4.24) and the requirement $F'^2 \neq G'^2$. Conversely, Eq. (4.25) implies $F' \neq -G'$ unless $F' = G' = 0$, which is not allowed. Also $F' = G'$ is not allowed because it would yield $e^{2\nu} = 0$. It follows that all class II solutions are obtained by solving Eq. (4.24) with the requirement $F'^2 \neq G'^2$. Let us introduce X and Y:

$$F \equiv \frac{1}{2}\,[X + Y], \quad G \equiv \frac{1}{2}\,[X - Y]\ . \tag{4.27}$$

Substitution into Eq. (4.24) yields

$$Y' = \pm\ 2\pi G_N \sigma\ C_0^{\ 1/2}\ \frac{X^{3/4}\ X'}{[1+(2\pi G_N\sigma)^2 C_0\ X^{3/2}]^{1/2}}\ . \tag{4.28}$$

Hence

$$Y = \pm\ (2\pi G_N \sigma C_0^{\ 1/2})^{-4/3}\ I[(2\pi G_N \sigma C_0^{\ 1/2})^{4/3}\ X] \tag{4.29}$$

where I(u) is the function

$$I(u) = \int \frac{u^{3/4}\ du}{(1+u^{3/2})^{1/2}}\ . \tag{4.30}$$

To assure $F'^2 \neq G'^2$, we now simply require $X' \neq 0$ and $C_0 \neq 0$.

In summary, the general class II domain wall solution has non-vanishing metric components

$$g_{xx} = g_{yy} = \frac{1}{2}\,[X(t-|z|) + Y(t-|z|) + X(t+|z|) - Y(t+|z|)]$$

$$-g_{tt} = g_{zz} = \frac{C_0}{2^{3/2}} \frac{[X'(t-|z|) + Y'(t-|z|)][X'(t+|z|) - Y'(t+|z|)]}{[X(t-|z|) + Y(t-|z|) + X(t+|z|) - Y(t+|z|)]^{1/2}} \quad (4.31)$$

where C_0 is an arbitrary constant, X is an arbitrary function satisfying $X' \neq 0$ everywhere and determined by the initial data, and Y is given in terms of X and C_0 by Eqs. (4.29) and (4.30).

d. Examination and Interpretation of a Particular Class I Solution

Of special interest to us is the sense in which the above solutions carry the attribute of repulsion implied by Eq. (2.15) or (4.15a). Rather than trying to examine this issue in a general way, we shall be content here to tackle it for a special solution within class I.

We focus on the solution (4.21) that depends on $t-|z|$ only, i.e. the solution for which K in Eq. (4.20) is a constant. For this "purely outgoing" solution then, solving Eq. (4.20) with K constant, we obtain

$$F(t-|z|) = \left(\frac{2}{K(4\pi G_N\sigma)^2}\right)^2 \frac{1}{(t-|z|)^2} \quad (4.32a)$$

$$e^{2\nu} = \frac{1}{(2\pi G\sigma)^2} \frac{1}{(t-|z|)^2} \quad (4.32b)$$

for an appropriate choice of the origin of time. The x and y coordinates can be rescaled [set $K = (4\pi G_N\sigma)^{-1}$] to bring the purely outgoing solution to the form

$$ds^2 = \frac{1}{(2\pi G_N\sigma)^2}\frac{1}{(t-|z|)^2}(-dt^2 + dz^2 + dx^2 + dy^2). \tag{4.33}$$

Note that this solution is valid only for $t-|z| < 0$ because of the requirement $F' > 0$. The solution (4.33) is conformally flat; but, since it is a vacuum solution off the wall, it must actually be flat there because both the Weyl and Ricci parts of the Riemann tensor vanish. The following coordinate transformation brings the metric (4.33) to Minkowski form:

$$t^* = \frac{1}{2}\left\{(t+z) - \frac{(2\pi G_N\sigma)^{-2}+x^2+y^2}{t-z}\right\},$$

$$z^* = \frac{1}{2}\left\{(t+z) + \frac{(2\pi G_N\sigma)^{-2}-(x^2+y^2)}{t-z}\right\},$$

$$x^* = \frac{-x}{(t-z)}\frac{1}{2\pi G_N\sigma}, \qquad y^* = \frac{-y}{(t-z)}\frac{1}{2\pi G_N\sigma}, \tag{4.34}$$

for $z > 0$. In terms of the new variables,

$$ds^2 = -dt^{*2} + dz^{*2} + dx^{*2} + dy^{*2}. \tag{4.35}$$

An analogous transformation to Minkowski coordinates exists for $z < 0$. The inverse of (4.34) is

$$t = \frac{1}{2}\left[t^* + z^* - \frac{(2\pi G_N\sigma)^{-2} + x^{*2} + y^{*2}}{t^* - z^*}\right],$$

$$z = \frac{1}{2}\left[t^* + z^* + \frac{(2\pi G_N\sigma)^{-2} - (x^{*2} + y^{*2})}{t^* - z^*}\right],$$

$$x = \frac{x^*}{(t^* - z^*)}\frac{1}{2\pi G_N\sigma}, \quad y = \frac{y^*}{(t^* - z^*)}\frac{1}{2\pi G_N\sigma}. \tag{4.36}$$

Since

$$t - z = -\frac{1}{(t^* - z^*)}\frac{1}{(2\pi G_N\sigma)^2}, \tag{4.37}$$

the allowed region $t - z < 0$ for $z \geqslant 0$ is mapped into the region $t^* - z^* > 0$ in the new coordinates. We now seek the location of the domain wall in the latter region.

Setting $z = 0$ in Eq. (4.36), we obtain

$$z^{*2} + x^{*2} + y^{*2} = \frac{1}{(2\pi G_N\sigma)^2} + t^{*2} \tag{4.38}$$

for the equation of the domain wall in the new coordinates. In the new coordinates, the domain wall is thus bent into a segment of a sphere defined by Eq. (4.38) and the coordinate restriction $t^* - z^* > 0$. If we analytically extend the solution (4.35) in the obvious way into the region $t^* - z^* < 0$, we pick up the remainder of the sphere which now completely encloses the original $z > 0$ side of the wall within its interior. In the Minkowski coordinates therefore, this "planar" domain wall is not a plane at all but rather an accelerating sphere. The sphere comes in from large distances, at near the speed of light in the far past; it has constant outward acceleration $2\pi G_N\sigma$, halts its collapse and reexpands. By reflection symmetry, it is clear that the $z < 0$ side of the wall in the original coordinates is also enclosed by an outwardly accelerating sphere in the Minkowski coordinates there. This behaviour is permitted on both sides because we have not demanded asymptotic flatness.

The locally repulsive nature of the domain wall is evident. Indeed, the exhibited motion of the wall in the Minkowski coordinate systems requires an observer riding with the wall to accelerate toward it, regardless of which side he is on, with acceleration $2\pi G_N \sigma$. Further, every geodesic observer sees the wall accelerating away from him with acceleration $2\pi G_N \sigma$.

V. CONCLUSIONS

We have found that domain walls have repulsive gravitational fields, as had been anticipated by Vilenkin [5]. Eq. (2.14b), which has general validity, states this result and expresses clearly what is to be understood thereby. The repulsive character of domain walls is also well illustrated by the exact solution (4.33) to Einstein's equations in the presence of a "planar" domain wall. For this solution, geodesic observers on either side are repelled by the wall with uniform "acceleration" $2\pi G_N\sigma$. "Practical" examples of "planar" domain walls with reflection symmetry that one might consider are a domain wall stretched by a static hoop and a domain wall stretched over the cosmological horizon. In the first case, a test particle placed next to the domain wall would be repelled by it, whereas a distant test particle, more than the hoop's diameter away, would be attracted by the domain wall-hoop system. Indeed, in an asymptotically flat space-time, everything (provided it has positive total energy) is gravitationally attractive from far away. This is true in particular of the spherical domain wall discussed in section III. Even though all domain walls are repulsive in the sense of Eq. (2.14b), a distant observer is attracted towards the center of a spherical domain wall. As was emphasized at the end of section III, these two statements are not contradictory because the spherical domain wall accelerates inward.

Next, let us consider a domain wall stretched over the cosmological horizon. It derives its stability from causality. If the domain wall is so close by that it traverses a region of the universe that we can observe sufficiently well today, its presence would be detected by the fact that it repells heavenly bodies on either side. Axion models [8-10], which have been proposed to explain the absence of P and CP violation in the strong interactions, have recently been shown to have domain walls [11,12]. In that case, the magnitude of the acceleration is

$$2\pi G_N \sigma \simeq 2\pi G_N \frac{8 m_a v^2}{N^2} \simeq 2\pi G_N \frac{4}{3} f_\pi m_\pi v \frac{1}{N} \simeq \frac{1}{10^5 \text{ sec}} \left(\frac{v}{10^{10} \text{ GeV}}\right) \frac{1}{N} \qquad (5.1)$$

where m_a is the axion mass, N is the number of vacua of the axion model [11,13] and v is the magnitude of the vacuum expectation value that breaks the $U_{PQ}(1)$ quasi-symmetry of Peccei and Quinn [8-10]. Astrophysical and cosmological constraints require v to lie in the range $10^8 \text{ GeV} \lesssim v \lesssim 10^{12} \text{ GeV}$ [14-15]. If, on the other hand, the domain wall stretched over the horizon is outside our presently observable universe, it may have escaped our notice because its gravitational field is one of constant acceleration and therefore does not produce any tidal effects.

Our results are also of relevance to the evolution of domain walls in the early universe, and to a discussion of the primordial density perturbations that such domain walls produce. The results of section III imply that a domain wall of size larger than $(4\pi G_N \sigma)^{-1}$ is a black hole. Closed domain walls of size less than $(4\pi G_N \sigma)^{-1}$ likely also collapse to black holes by radiating away their asphericity. The collapse of closed domain walls would provide us with a new source of primordial black holes which may find their way into the halos of galaxies. In axion models, the domain walls appear when the universe has cooled to a temperature $T \sim 1$ GeV. If N=1 (that is to say, if the axion model has a unique vacuum), the domain walls have initial size of order 10^{-4} sec, the age of the universe at that time [12,13]. These domain walls are of both the open and closed variety. The black holes produced by the collapse of the closed domain walls would have mass of order

$$M \sim 4\pi\sigma R_I^2\Big|_{R_I \sim 10^{-4}\ \mathrm{sec}} \sim 10^{-8} M_0 \left(\frac{v}{10^{10}\mathrm{GeV}}\right) . \qquad (5.2)$$

If black holes of such characteristic mass have found their way into galactic halos, they probably would have gone undetected [16].

If the axion model has multiply degenerate vacua ($N > 1$) [11], causality implies that there is at least of order one domain wall per horizon at any time after $t \approx 10^{-4}$ sec. This is because causally disconnected regions of the universe will in general be in different vacua. With one domain wall per horizon, the universe's energy density becomes domain wall dominated at cosmological time

$$t_{d.w.} \sim \frac{1}{32\pi G_N \sigma} \approx .6 \times 10^4 \ \mathrm{sec} \left(\frac{10^{10}\mathrm{GeV}}{v}\right) N \qquad (5.3)$$

After $t_{d.w.}$ the large scale expansion of the universe goes like $R \sim t^2$, where R is the cosmological scale parameter [3]. Here "large scale" means length scales comprising many domain bubbles. One might question whether such a domain wall dominated universe is in disagreement with observation since our presently observable universe would be completely inside a domain bubble, far from any domain walls. The trouble, however, is that the amount of matter in our neighborhood of the universe is that same amount which was contained in our domain bubble at time $t_{d.w.}$, that is to say, much less than what we observe today.

Finally, it may be interesting to consider theories in which there is a near perfect degeneracy of the vacuum, but with a very slight breaking of the degeneracy present [11]. In that case the domain walls disappear after a time of order

$$t_B \simeq \frac{\sigma}{\Delta \mathcal{H}} \tag{5.4}$$

provided $t_B < t_{d.w.}$ and provided the decay time, in which a disconnected domain wall within its horizon can radiate itself away, is smaller than t_B. In (5.4), $\Delta \mathcal{H}$ is the energy density difference between the almost degenerate vacua. t_B is the size of domains for which the differences in volume energy are of order the surface energy. Once the domain walls have average size t_B, the degeneracy breaking effects become important and the true vacuum takes over. If t_B is sufficiently large, the density perturbations produced by the domain walls before they disappear may become the seed masses from which galaxies evolved in a hierarchical clustering model of galaxy formation. The hierarchical clustering model is the appropriate one [17] to an axion dominated universe [15].

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